BOUNDARY VALUE PROBLEMS FOR THE 2nd-ORDER SEIBERG-WITTEN EQUATIONS

DR. CELSO M DORIA

ABSTRACT. It is shown that the non-homogeneous Dirichlet and Neuman problems for the 2nd-order Seiberg-Witten equation admit a regular solution once the $\mathcal{H}$-condition 3.1.1 is satisfied. The approach consist in applying the elliptic techniques to the variational setting of the Seiberg-Witten equation.

1. Introduction

Let X be a compact smooth 4-manifold with non-empty boundary. In our context, the Seiberg-Witten equations are the 2nd-order Euler-Lagrange equation of the functional defined in 2.2.1. When the boundary is empty, their variational aspects were first studied in [9] and the topological ones in [2]. Thus, the main aim is to obtain the existence of a solution to the non-homogeneous equations whenever $\partial X \neq \emptyset$. The non-emptyness of the boundary inflicts boundary conditions on the problem. Classicaly, this sort of problem is classified according with its boundary conditions in *Dirichlet Problem* ($\mathcal{D}$) or *Neumann Problem* ($\mathcal{N}$).

1.1. $Spin^c$ Structure.
The space of $Spin^c$ structures on X is identified with

$$Spin^c(X) = \{\alpha + \beta \in H^2(X, \mathbb{Z}) \oplus H^1(X, \mathbb{Z}_2) \mid w_2(X) = \alpha(mod\ 2)\}.$$

For each $\alpha \in Spin^c(X)$, there is a representation $\rho_\alpha : SO_4 \to \mathbb{C}l_4$, induced by a $Spin^c$ representation, and a pair of vector bundles $(\mathcal{S}_\alpha{}^+, \mathcal{L}_\alpha)$ over X (see [11]). Let P_{SO_4} be the frame bundle of X, so

- $\mathcal{S}_\alpha = P_{SO_4} \times_{\rho_\alpha} V = \mathcal{S}_\alpha{}^+ \oplus \mathcal{S}_\alpha{}^-$.
  The bundle $\mathcal{S}_\alpha{}^+$ is the positive complex spinors bundle (fibers are $Spin_4^c - modules$ isomorphic to $\mathbb{C}^2$)
- $\mathcal{L}_\alpha = P_{SO_4} \times_{det(\alpha)} \mathbb{C}$.
  It is called the *determinant line bundle* associated to the $Spin^c$-struture α. $(c_1(\mathcal{L}_\alpha) = \alpha)$

Thus, for each $\alpha \in Spin^c(X)$ we associate a pair of bundles

$$\alpha \in Spin^c(X) \quad \leadsto \quad (\mathcal{L}_\alpha, \mathcal{S}_\alpha{}^+).$$

From now on, we considered on X a Riemannian metric g and on $\mathcal{S}_\alpha$ an hermitian structure h.

Let P_α be the U_1-principal bundle over X obtained as the frame bundle of $\mathcal{L}_\alpha$ $(c_1(P_\alpha) = \alpha)$. Also, we consider the adjoint bundles

$$Ad(U_1) = P_{U_1} \times_{Ad} U_1 \quad ad(\mathfrak{u}_1) = P_{U_1} \times_{ad} \mathfrak{u}_1,$$

where $Ad(U_1)$ is a fiber bundle with fiber U_1, and $ad(\mathfrak{u}_1)$ is a vector bundle with fiber isomorphic to the Lie Algebra $\mathfrak{u}_1$.

1.2. **The Main Theorem.** Let $\mathcal{A}_\alpha$ be (formally) the space of connections (covariant derivative) on $\mathcal{L}_\alpha$, $\Gamma(\mathcal{S}_\alpha^+)$ is the space of sections of $\mathcal{S}_\alpha^+$ and $\mathcal{G}_\alpha = \Gamma(Ad(U_1))$ is the gauge group acting on $\mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+)$ as follows:

$$(1.1) \qquad\qquad\qquad g.(A,\phi) = (A + g^{-1}dg, g^{-1}\phi).$$

$\mathcal{A}_\alpha$ is an afim space which vector space structure, after fixing an origin, is isomorphic to the space $\Omega^1(ad(\mathfrak{u}_1))$ of $ad(\mathfrak{u}_1)$-valued 1-forms. Once a connection $\nabla^0 \in \mathcal{A}_\alpha$ is fixed, a bijection $\mathcal{A}_\alpha \leftrightarrow \Omega^1(ad(\mathfrak{u}_1))$ is explicited by $\nabla^A \leftrightarrow A$, where $\nabla^A = \nabla^0 + A$. $\mathcal{G}_\alpha = Map(X, U_1)$, since $Ad(U_1) \simeq X \times U_1$. The curvature of a 1-connection form $A \in \Omega^1(ad(\mathfrak{u}_1))$ is the 2-form $F_A = dA \in \Omega^2(ad(\mathfrak{u}_1))$.

Definition 1.2.1.   (1) *the configuration space of the $\mathcal{D}$-problem is*

$$(1.2) \qquad\quad \mathcal{C}_\alpha^\mathcal{D} = \{(A,\phi) \in \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+) \mid (A,\phi)\mid_Y \overset{gauge}{\sim} (A_0, \phi_0)\},$$

(2) *the configuration space of the $\mathcal{N}$-problem is*

$$(1.3) \qquad\qquad\qquad \mathcal{C}_\alpha^\mathcal{N} = \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+)$$

Although each boundary problem requires its own configuration space, the superscripts $\mathcal{D}$ and $\mathcal{N}$ will be used whenever the distintion is necessary, since most arguments works for both sort of problems.

The Gauge Group $\mathcal{G}_\alpha$ action on each of the configuration space is given by 1.1.

The Dirichlet ($\mathcal{D}$) and Neumann ($\mathcal{N}$) boundary value problems associated to the $\mathcal{SW}_\alpha$-equations are the following: Let's consider $(\Theta, \sigma) \in \Omega^1(ad(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$ and (A_0, ϕ_0) defined on the manifold ∂X (A_0 is a connection on $\mathcal{L}_\alpha \mid_{\partial X}$, ϕ_0 is a section of $\Gamma(\mathcal{S}_\alpha^+ \mid_{\partial X})$). In this way, find $(A,\phi) \in \mathcal{C}_\alpha^\mathcal{D}$ satisfying $\mathcal{D}$ and $(A,\phi) \in \mathcal{C}_\alpha^\mathcal{N}$ satisfying $\mathcal{N}$, where

$$(1.4) \qquad \mathcal{D} = \begin{cases} d^*F_A + 4\Phi^*(\nabla^A\phi) = \Theta, \\ \Delta_A\phi + \frac{(|\phi|^2 + k_g)}{4}\phi = \sigma, \\ (A,\phi)\mid_{\partial X} \overset{gauge}{\sim} (A_0, \phi_0), \end{cases} \qquad \mathcal{N} = \begin{cases} d^*F_A + 4\Phi^*(\nabla^A\phi) = \Theta, \\ \Delta_A\phi + \frac{(|\phi|^2 + k_g)}{4}\phi = \sigma, \\ i^*(*F_A) = 0, \nabla_\nu^A\phi = 0, \end{cases}$$

and

(1) the operator $\Phi^* : \Omega^1(\mathcal{S}_\alpha{}^+) \to \Omega^1(\mathfrak{u}_1)$ is locally given by

$$(1.5) \qquad\qquad \Phi^*(\nabla^A\phi) = \frac{1}{2}\nabla^A(|\phi|^2) = \sum_i <\nabla_i^A\phi, \phi> \eta_i,$$

and $\eta = \{\eta_i\}$ is an orthonormal frame in $\Omega^1(ad(\mathfrak{u}_1))$.

(2) $i^*(*F_A) = F_4$, where
$F_4 = (F_{14}, F_{24}, F_{34}, 0)$ is the local representation of the 4th-component (normal to ∂X) of the 2-form of curvature in the local chart (x, U) of X;
$x(U) = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4; \| x \| < \epsilon, x_4 \geq 0\}$, and

$x(U \cap \partial X) \subset \{x \in x(U) \mid x_4 = 0\}$. Let $\{e_1, e_2, e_3, e_4\}$ be the canonical base of $\mathbb{R}^4$, so $\nu = -e_4$ is the normal vector field along ∂X.

Main Theorem 1.2.2. *If the pair $(\Theta, \sigma) \in L^{k,2} \oplus (L^{k,2} \cap L^\infty)$ satisfies the $\mathcal{H}$-condition 3.1.1, then the problems $\mathcal{D}$ and $\mathcal{N}$ admit a C^r-regular solution (A, ϕ), whenever $2 < k$ and $r < k$.*

2. **Basic Set Up**

2.1. **Sobolev Spaces.** As a vector bundle E over (X,g) is endowed with a metric and a covariant derivative ∇, we define the Sobolev norm of a section $\phi \in \Omega^0(E)$ as

$$\| \phi \|_{L^{k,p}} = \sum_{|i|=0}^{k} \left(\int_X | \nabla^i \phi |^p \right)^{\frac{1}{p}}.$$

In this way, the $L^{k,p}$-Sobolev Spaces of sections of E is defined as

$$L^{k,p}(E) = \{\phi \in \Omega^0(E) \mid \quad \| \phi \|_{L^{k,p}} < \infty\}.$$

In our context, in which we fixed a connection ∇^0 on $\mathcal{L}_\alpha$, a metric g on X and an hermitian structure on $\mathcal{S}_\alpha$, the Sobolev Spaces on which the basic setting is made are the following;

- $\mathcal{A}_\alpha = L^{1,2}(\Omega^1(ad(\mathfrak{u}_1)))$;
- $\Gamma(\mathcal{S}_\alpha{}^+) = L^{1,2}(\Omega^0(X, \mathcal{S}_\alpha{}^+))$;
- $\mathcal{C}_\alpha = \mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha{}^+)$;
- $\mathcal{G}_\alpha = L^{2,2}(X, U_1) = L^{2,2}(Map(X, U_1))$.
  ($\mathcal{G}_\alpha$ is an ∞-dimensional Lie Group which Lie algebra is $\mathfrak{g} = L^{1,2}(X, \mathfrak{u}_1)$).

The Sobolev spaces above induce a Sobolev structure on $\mathcal{C}_\alpha^{\mathcal{D}}$ and on $\mathcal{C}_\alpha^{\mathcal{N}}$. From now on, the configuration spaces will be denoted by $\mathcal{C}_\alpha$ by ignoring the superscripts, unless if it needed be.

The most basic analytical results needed to achieve the main result is the *Gauge Fixing Lemma* (Uhlenbeck - [15]) and the estimate 2.1, both extended by Marini,A. [12] to manifolds with boundary;

Lemma 2.1.1. *(Gauge Fixing Lemma) - Every connection $\hat{A} \in \mathcal{A}_\alpha$ is gauge equivalent, by a gauge transformation $g \in \mathcal{G}_\alpha$ named Coulomb ($\mathfrak{C}$) gauge, to a connection $A \in \mathcal{A}_\alpha$ satisfying*

(1) $d_\tau^{*f} A_\tau = 0$ *on ∂X,*
(2) $d^* A = 0$ *on X.*
(3) *In the $\mathcal{N}$-problem, the connection A satisfies $A_\nu = 0$ ($\nu \perp \partial X$).*

Corollary 2.1.2. *Under the hypothesis of 2.1.1, there exists a constant $K > 0$ such that the connection A, gauge equivalent to $\hat{A}$ by the Coulomb gauge, satisfies the following estimates:*

(2.1) $$\| A \|_{L^{1,p}} \leq K. \| F_A \|_{L^p}$$

notation: $*_f$ is the Hodge operator in the flat metric and the index τ denotes tangential components.

2.2. **Variational Formulation.** A global formulation for problems $\mathcal{D}$ and $\mathcal{N}$ is made using the Seiberg-Witten functional;

Definition 2.2.1. *Let $\alpha \in Spin^c(X)$. The Seiberg-Witten functional $\mathcal{SW}_\alpha : \mathcal{C}_\alpha \to \mathbb{R}$ is defined as*

$$(2.2) \quad \mathcal{SW}_\alpha(A, \phi) = \int_X \{\frac{1}{4} \mid F_A \mid^2 + \mid \nabla^A \phi \mid^2 + \frac{1}{8} \mid \phi \mid^4 + \frac{k_g}{4} \mid \phi \mid^2 \} dv_g + \pi^2 \alpha^2.$$

where k_g= scalar curvature of (X,g).

Remark 2.2.2. *The $\mathcal{G}_\alpha$-action on $\mathcal{C}_\alpha$ has the following properties;*
  (1) *the $\mathcal{SW}_\alpha$-functional is $\mathcal{G}_\alpha$-invariant.*
  (2) *the $\mathcal{G}_\alpha$-action on $\mathcal{C}_\alpha$ induces on $T\mathcal{C}_\alpha$ a $\mathcal{G}_\alpha$-action as follows:*
      let $(\Lambda, V) \in T_{(A,\phi)}\mathcal{C}_\alpha$ and $g \in \mathcal{G}_\alpha$,

$$g.(\Lambda, V) = (\Lambda, g^{-1}V) \in T_{g.(A,\phi)}\mathcal{C}_\alpha.$$

  Consequently, $d(\mathcal{SW}_\alpha)_{g.(A,\phi)}(g.(\Lambda, V)) = d(\mathcal{SW}_\alpha)_{(A,\phi)}(\Lambda, V)$.

The tangent bundle $T\mathcal{C}_\alpha$ decomposes as

$$T\mathcal{C}_\alpha = \Omega^1(ad(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+).$$

In this way, the 1-form $d\mathcal{SW}_\alpha \in \Omega^1(\mathcal{C}_\alpha)$ admits a decomposition $d\mathcal{SW}_\alpha = d_1\mathcal{SW}_\alpha + d_2\mathcal{SW}_\alpha$, where

$$d_1(\mathcal{SW}_\alpha)_{(A,\phi)} : \Omega^1(ad(\mathfrak{u}_1)) \to \mathbb{R}, \quad d_1(\mathcal{SW}_\alpha)_{(A,\phi)}.\Lambda = d(\mathcal{SW}_\alpha)_{(A,\phi)}.(\Lambda, 0)$$

$$d_2(\mathcal{SW}_\alpha)_{(A,\phi)} : \Gamma(\mathcal{S}_\alpha^+) \to \mathbb{R}, \quad d_2(\mathcal{SW}_\alpha)_{(A,\phi)}.V = d(\mathcal{SW}_\alpha)_{(A,\phi)}.(0, V).$$

By performing the computations, we get
  (1) for every $\Lambda \in \mathcal{A}_\alpha$,

$$(2.3) \qquad d_1(\mathcal{SW}_\alpha)_{(A,\phi)}.\Lambda = \frac{1}{4} \int_X Re\{< F_A, d_A\Lambda > + 4 < \nabla^A(\phi), \Phi(\Lambda) >\}dx,$$

   where $\Phi : \Omega^1(\mathfrak{u}_1) \to \Omega^1(\mathcal{S}_\alpha{}^+)$ is the linear operator $\Phi(\Lambda) = \Lambda(\phi)$, which dual is defined in 1.5,
  (2) for every $V \in \Gamma(\mathcal{S}_\alpha^+)$,

$$(2.4) \qquad d_2(\mathcal{SW}_\alpha)_{(A,\phi)}.V = \int_X Re\{< \nabla^A\phi, \nabla^A V > + < \frac{\mid \phi \mid^2 + k_g}{4}\phi, V >\}dx.$$

Therefore, by taking $supp(\Lambda) \subset int(X)$ and $supp(V) \subset int(X)$, we restrict to the interior of X, and so, the gradient of the $\mathcal{SW}_\alpha$-functional at $(A, \phi) \in \mathcal{C}_\alpha$ is

$$(2.5) \qquad grad(\mathcal{SW}_\alpha)(A, \phi) = (d_A^* F_A + 4\Phi^*(\nabla^A\phi), \triangle_A\phi + \frac{\mid \phi \mid^2 + k_g}{4}\phi)$$

It follows from the $\mathcal{G}_\alpha$-action on $T\mathcal{C}_\alpha$ that

$$(2.6) \quad grad(\mathcal{SW}_\alpha)(g.(A,\phi)) = \left(d_A^* F_A + 4\Phi^*(\triangledown^A \phi), g^{-1}.(\triangle_A \phi + \frac{\mid \phi \mid^2 + k_g}{4}\phi) \right).$$

An important analytical aspect of the $\mathcal{SW}_\alpha$-functional is the Coercivity Lemma proved in [9];

Lemma 2.2.3. Coercivity - *For each $(A,\phi) \in \mathcal{C}_\alpha$, there exists $g \in \mathcal{G}_\alpha$ and a constant $K_{\mathfrak{C}}^{(A,\phi)} > 0$, where $K_{\mathfrak{C}}^{(A,\phi)}$ depends on (X,g) and $\mathcal{SW}_\alpha(A,\phi)$, such that*

$$\| g.(A,\phi) \|_{L^{1,2}} < K_{\mathfrak{C}}^{(A,\phi)}.$$

Proof. lemma 2.3 in [9]. The gauge transform is the Coulomb one given in the Gauge Fixing Lemma 2.1.1. □

Considering the gauge invariance of the $\mathcal{SW}_\alpha$-theory, and the fact that the gauge group $\mathcal{G}_\alpha$ is a infinite dimensional Lie Group, we can't hope to handle the problem in the general. So forth, we need to restrict the problem to the space

$$(2.7) \qquad \mathcal{C}_\alpha^{\mathfrak{C}} = \{(A,\phi) \in \mathcal{C}_\alpha; \| (A,\phi) \|_{L^{1,2}} < K_{\mathfrak{C}}^{(A,\phi)}\},$$

The superscript $\mathcal{D}$ and $\mathcal{N}$ are being ignored for simplicity, although each one should be taken in account according with the problem. These choice of spaces is a a property of the $\mathcal{G}_\alpha$ action on $\mathcal{C}_\alpha$, it is suggested by the Gauge Fixing Lemma and the Coercivity Lemma; this sort of propertie is not shared by most actions.

3. **Existence of a Solution**

3.1. **Non Homogeneous Palais-Smale Condition - $\mathcal{H}$.** -

In the variational formulation, the problems $\mathcal{D}$ and $\mathcal{N}$ (1.4) are written as

$$(3.1)$$
$$(\mathcal{D}) = \begin{cases} grad(\mathcal{SW}_\alpha)(A,\phi) = (\Theta,\sigma), \\ (A,\phi) \mid_{\partial X} \overset{\text{gauge}}{\sim} (A_0,\phi_0). \end{cases} \quad (\mathcal{N}) = \begin{cases} grad(\mathcal{SW}_\alpha)(A,\phi) = (\Theta,\sigma), \\ i^*(*F_A) = 0, \nabla_n^A \phi = 0, \end{cases}$$

The equations in 1.4 may not admit a solution for any pair $(\Theta,\sigma) \in \Omega^1(ad(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$. In finite dimension, if we consider a function $f: X \to \mathbb{R}$, the analogous question would be to find a point $p \in X$ such that, for a fixed vector u, $grad(f)(p) = u$. This question is more subtle if f is invariant by a Lie group action on X. Therefore, we need a premiss on the pair $(\Theta,\sigma) \in \Omega^1(ad(\mathfrak{u}_1)) \oplus \Gamma(\mathcal{S}_\alpha^+)$;

Condition 3.1.1. ($\mathcal{H}$) - *Let $(\Theta,\sigma) \in L^{1,2}(\Omega^1(ad(\mathfrak{u}_1))) \oplus (L^{1,2}(\Gamma(\mathcal{S}_\alpha^+)) \cap L^\infty(\Gamma(\mathcal{S}_\alpha^+)))$ be a pair such that there exists a sequence $\{(A_n,\phi_n)\}_{n \in \mathbb{Z}} \subset \mathcal{C}_\alpha^{\mathfrak{C}}$ (2.7) with the following properties;*

1. *$\{(A_n,\phi_n)\}_{n \in \mathbb{Z}} \subset L^{1,2}(\mathcal{A}_\alpha) \times (L^{1,2}(\Gamma(\mathcal{S}_\alpha^+)) \cup L^\infty(\Gamma(\mathcal{S}_\alpha^+)))$ and there exists a constant $c_\infty > 0$ such that, for all $n \in \mathbb{Z}$, $\| \phi_n \|_\infty < c_\infty$.*
2. *there exists $c \in \mathbb{R}$ such that, for all $n \in \mathbb{Z}$, $\mathcal{SW}_\alpha(A_n,\phi_n) < c$,*

(3) *the sequence* $\{d(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}\}_{n\in\mathbb{Z}} \subset \left(L^{1,2}(\Omega^1(ad(\mathfrak{u}_1))) \oplus L^{1,2}(\Gamma(\mathcal{S}_\alpha^+))\right)^*$,
of linear functionals, converges weakly to

$$L_\Theta + L_\sigma : T\mathcal{C}_\alpha \to \mathbb{R},$$

where

$$L_\Theta(\Lambda) = \int_X < \Theta, \Lambda >, \quad L_\sigma(V) = \int_X < \sigma, V > .$$

3.2. **Strong Converge of** $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$ **in** $L^{1,2}$. -
As an immediate consequency of (2.2.3), the sequence $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$ given by
the $\mathcal{H}$-condition converges to a pair (A, ϕ);

(1) weakly in $\mathcal{C}_\alpha$,
(2) weakly in $L^4(\mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+))$,
(3) strongly in $L^p(\mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+))$, for any $p < 4$.

Remark 3.2.1. *Let* $\{A_n\}_{n\in\mathbb{N}} \subset L^2$ *be a converging sequence in* L^2 *satisfying* $d^*A_n = 0$, *for all* $n \in \mathbb{N}$, *and let* $A = \lim_{n\to\infty} A_n \in L^2$. *So,* $d^*A = 0$, *once*

$$|< d^*A, \rho >| \leq | A - A_n |_{L^2} . | d\rho |_{L^2},$$

for all $\rho \in \Omega^0(ad(\mathfrak{u}_1))$.

Theorem 3.2.2. A - *The limit* $(A, \phi) \in L^2(\mathcal{A}_\alpha \times \Gamma(\mathcal{S}_\alpha^+))$, *obtained as a limit of the sequence* $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$, *is a weak solution of 1.4.*

Proof. The proof goes along the same lines as in the 2nd-step in the proof of the
Main Theorem in [9].

(1) for every $\Lambda \in \mathcal{A}_\alpha$,

$$(3.2) \quad d_1(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}.\Lambda = \frac{1}{4}\int_X Re\{< F_{A_n}, d_{A_n}\Lambda > +4 < \nabla^{A_n}(\phi_n), \Phi(\Lambda) >\}dx$$

$$(3.3) \qquad\qquad + \int_{\partial X} Re\{\Lambda \wedge *F_{A_n}\}$$

where
(a) $\Phi : \Omega^1(\mathfrak{u}_1) \to \Omega^1(\mathcal{S}_\alpha{}^+)$ is the linear operator $\Phi(\Lambda) = \Lambda(\phi)$; it's dual
is defined in 1.5,
Assuming $\phi \in L^\infty$ (3.2.3), it follows that

$$\lim_{n\to\infty} d_1(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}.\Lambda = d_1(\mathcal{SW}_\alpha)_{(A,\phi)}.\Lambda.$$

Therefore, $d_1(\mathcal{SW}_\alpha)_{(A,\phi)}.\Lambda = \int_X < \Theta, \Lambda >$.
(b) $\Lambda \wedge *F_A = - < \Lambda, F_4 > dx_1 \wedge dx_2 \wedge dx_3$.
Since the equation above is true for all Λ, let $supp(\Lambda) \subset \partial X$, so $F_4 = 0$
$(\Rightarrow i^*(*F_A) = 0)$.

(2) for every $V \in \Gamma(\mathcal{S}_\alpha^+)$,

(3.4)
$$d_2(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}.V = \int_X Re\{<\nabla^{A_n}\phi_n, \nabla^{A_n}V>\} + <\frac{|\phi_n|^2 + k_g}{4}\phi_n, V>\}dx$$

(3.5)
$$+ \int_{\partial X} Re\{<\nabla_\nu^{A_n}\phi_n, V>\}.$$

Analougously, it follows that (A, ϕ) is a weak solution of the equation

$$d_2(\mathcal{SW}_\alpha)_{(A,\phi)}.V = \int_X <\sigma, V>.$$

So, in the $\mathcal{N}$-problem, $\nabla_\nu^A\phi = 0$.

$\square$

In order to pursue the strong $L^{1,2}$-convergence for the sequence $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$, next we obtain an upper bound for $\| \phi \|_{L^\infty}$, whenever (A, ϕ) is a weak solution .

Lemma 3.2.3. *Let (A, ϕ) be a solution of either $\mathcal{D}$ or $\mathcal{N}$ in 1.4, so*

(1) *If $\sigma = 0$, then there exists a constant $k_{X,g}$, depending on the Riemannian metric on X, such that*

(3.6)
$$\| \phi \|_\infty < k_{X,g}vol(X).$$

(2) *If $\sigma \neq 0$, then there exist constantc $c_1 = c_1(X, g)$ and $c_2 = c_2(X, g)$ such that*

(3.7)
$$\| \phi \|_{L^p} < c_1 + c_2 \| \sigma \|_{L^{3p}}^3 .$$

In particular, if $\sigma \in L^\infty$ then $\phi \in L^\infty$

Proof. Fix $r \in \mathbb{R}$ and suppose that there is a ball $B_{r^{-1}}(x_0)$, around the point $x_0 \in X$, such that

$$|\phi(x)| > r, \quad \forall x \in B_{r^{-1}}(x_0).$$

Define

$$\eta = \begin{cases} \left(1 - \frac{r}{|\phi|}\right)\phi, & \text{if } x \in B_{r^{-1}}(x_0), \\ 0, & \text{if } x \in X - B_{r^{-1}}(x_0) \end{cases}$$

So,

(3.8)
$$|\eta| \leq |\phi|$$

$$\nabla\eta = r\frac{<\phi, \nabla\phi>}{|\phi|^3}\phi + (1 - \frac{r}{|\phi|})\nabla\phi$$

$$\Rightarrow |\nabla\eta|^2 = r^2\frac{<\phi, \nabla\phi>^2}{|\phi|^4} + 2r(1 - \frac{r}{|\phi|})\frac{<\phi, \nabla\phi>^2}{|\phi|^3} + (1 - \frac{r}{|\phi|})^2 |\nabla\phi|^2$$

$$\Rightarrow |\nabla\eta|^2 < r^2\frac{|\nabla\phi|^2}{|\phi|^2} + 2r(1 - \frac{r}{|\phi|})\frac{|\nabla\phi|^2}{|\phi|} + (1 - \frac{r}{|\phi|})^2 |\nabla\phi|^2 .$$

Since $r < | \phi |$,

(3.9) $$| \nabla \eta |^2 < 4 | \nabla \phi |^2 .$$

Hence, by 3.8 and 3.9, $\eta \in L^{1,2}$.

The directional derivative of $\mathcal{SW}_\alpha$ at direction η is given by

$$d(\mathcal{SW}_\alpha)_{(A,\phi)}(0, \eta) = \int_X [< \nabla^A \phi, \nabla^A \eta > + \frac{| \phi |^2 + k_g}{4} | \phi | (| \phi | - r)].$$

By 2.4),

$$\int_X [< \nabla^A \phi, \nabla^A \eta > + \frac{| \phi |^2 + k_g}{4} | \phi | (| \phi | - r)] = \int_X < \sigma, (1 - \frac{r}{| \phi |})\phi > .$$

However,

$$\int_X < \nabla^A \phi, \nabla^A \eta > = \int_X [r \frac{< \phi, \nabla^A \phi >^2}{| \phi |^3} + (1 - \frac{r}{| \phi |}) | \nabla \phi |^2] > 0.$$

So,

$$\int_X \frac{| \phi |^2 + k_g}{4} | \phi | (| \phi | - r) < \int_X < \sigma, (1 - \frac{r}{| \phi |})\phi > < \int_X | \sigma | (| \phi | - r).$$

Hence,

$$\int_X (| \phi | - r) \left(\frac{| \phi |^2 + k_g}{4} | \phi | - | \sigma | \right) < 0.$$

Since $r < | \phi(x) |$, whenever $x \in B_{r^{-1}}(x_0)$, it follows that

(3.10) $(| \phi |^2 + k_g) | \phi | < 4 | \sigma |,$ almost everywhere in $B_{r^{-1}}(x_0)$.

There are two cases to be analysed independently;

(1) $\sigma = 0$.
    In this case, we get

(3.11) $(| \phi |^2 + k_g) | \phi | < 0,$ almost everywhere.

    The scalar curvature plays a central role here: if $k_g \geq 0$ then $\phi = 0$; otherwise,

$$| \phi | \leq max\{0, (-k_g)^{1/2}\}.$$

    Since X is compact, we let $k_{X,g} = \max_{x \in X}\{0, [-k_g(x)]^{1/2}\}$, and so,

$$|| \phi ||_\infty < k_{X,g} vol(X).$$

(2) Let $\sigma \neq 0$.
    The inequality 3.10 implies that

$$| \phi |^3 + k_g | \phi | - 4 | \sigma | < 0   \text{a.e.}$$

    Consider the polynomial

$$Q_{\sigma(x)}(w) = w^3 + k_g w - 4 \mid \sigma(x) \mid .$$

A estimate for $\mid \phi \mid$ is obtained by estimating the largest real number w satisfying $Q_{\sigma(x)}(w) < 0$. $Q_{\sigma(x)}$ being monic implies that $\lim_{w \to \infty} Q_{\sigma(x)}(w) = +\infty$. So, either $Q_{\sigma(x)} > 0$, whenever $w > 0$, or there exist a root $\rho \in (0, \infty)$. The first case would imply that

$$Q_{\sigma(x)}(\mid \phi(x) \mid) > 0, \quad a.e.,$$

contradicting 3.10. By the same argument, there exists a root $\rho \in (0, \infty)$ such that $Q_{\sigma(x)}(w)$ chances its sign in a neighboorhood of ρ. Let ρ be the largest root in $(0, \infty)$ with this propertie. By the Corollary A.0.11, there exist constants $c_1 = c_1(X, g)$ and c_2 such that

$$\mid \rho \mid < c_1 + c_2 \mid \sigma(x) \mid^3 .$$

Consequently,

$$(3.12) \qquad \mid \phi(x) \mid < c_1 + c_2 \mid \sigma(x) \mid^3, \quad \text{a.e. in } B_{r^{-1}}(x_0)$$

and

$$(3.13) \qquad \parallel \phi \parallel_{L^p} < C_1 + C_2 \parallel \sigma \parallel_{L^{3p}}^3, \quad \text{restricted to } B_{r^{-1}}(x_0)$$

where C_1, C_2 are constants depending on $vol(B_{r^{-1}}(x_0))$.
The inequality 3.13 can be extended over X by using a C^∞ partition of unity. Moreover, if $\sigma \in L^\infty$, then

$$(3.14) \qquad \parallel \phi \parallel_\infty < C_1 + C_2 \parallel \sigma \parallel_\infty^3,$$

where C_1, C_2 are constants depending on $vol(X)$.

$\square$

In [9], it was proved a sort of concentration lemma, which is extended as follows;

Lemma 3.2.4. *Let $\{(A_n, \phi_n)\}_{n \in \mathbb{Z}}$ be the sequence given by the $\mathcal{H}$-condition 3.1.1. So,*

$$\lim_{n \to \infty} \int_X < \Phi^*(\nabla^{A_n} \phi_n), A_n - A >= 0.$$

Proof. By equation 1.5,

$$\lim_{n \to \infty} \int_X < \Phi^*(\nabla^{A_n} \phi_n), A_n - A >= \lim_{n \to \infty} \int_X < \nabla_i^{A_n} \phi_n, \phi_n > . < \eta_i, A_n - A >$$

$$\lim_{n\to\infty} \int_X < \nabla_i^{A_n}\phi_n, \phi_n > . < \eta_i, A_n - A > \ \le$$

$$\lim_{n\to\infty} \int_X |< \nabla_i^{A_n}\phi_n, \phi_n >|^2 . \int_X |< \eta_i, A_n - A >|^2 \ \le$$

$$\lim_{n\to\infty} \left[\int_X | \nabla_i^{A_n}\phi_n |^2 . | \phi_n |^2 \right] . \int_X | A_n - A |^2 \ \le$$

$$\lim_{n\to\infty} c_\infty. \left[\int_X | \nabla_i^{A_n}\phi_n |^2 \right] . \| A_n - A \|_{L^2}^2 \ \le$$

$$\lim_{n\to\infty} c_\infty. \| \phi_n \|_{L^{1,2}}^2 . \| A_n - A \|_{L^2}^2 = 0.$$

$\square$

Theorem 3.2.5. B - *Let (Θ, σ) be a pair satisfying the $\mathcal{H}$- condition 3.1.1. So, the sequence $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$, given by 3.1.1, converges strongly to $(A, \phi) \in \mathcal{C}_\alpha$.*

Proof. From 3.2.2, $\{(A_n, \phi_n)\}_{n\in\mathbb{Z}}$ converges weakly in $L^{1,2}$ to $(A, \phi) \in \mathcal{C}_\alpha$. The prove is splitted into 2 parts;

(1) $\lim_{n\to\infty} \| A_n - A \|_{L^{1,2}} = 0$.
    Let $d^* : \Omega^1(ad(\mathfrak{u}_1)) \to \Omega^0(ad(\mathfrak{u}_1))$. The operator $d : ker(d^*) \to \Omega^2(ad(\mathfrak{u}_1))$ being elliptic implies, by the fundamentalelliptic estimate, that

$$\| A_n - A \|_{L^{1,2}} \le c \| d(A_n - A) \|_{L^2} + \| A_n - A \|_{L^2} .$$

The first term in the right-hand-side is estimate as follows;

$$\| dA_n - dA \|_{L^2}^2 = \int_X < d(A_n - A), d(A_n - A) >=$$

$$= \int_X < dA_n, d(A_n - A) > - \int_X < dA, d(A_n - A) >=$$

$$= \int_X < d^*F_{A_n}, A_n - A > - \int_X < d^*F_A, A_n - A >=$$

$$= \ d(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}(A_n - A) - 4 \int_X < \Phi^*(\nabla^{A_n}\phi_n), A_n - A > -$$

$$d(\mathcal{SW}_\alpha)_{(A,\phi)}(A_n - A) - 4 \int_X < \Phi^*(\nabla^A\phi), A_n - A > +o(1)$$

$$= -4 \left\{ \int_X < \Phi^*(\nabla^{A_n}\phi_n), A_n - A > + \int_X < \Phi^*(\nabla^A\phi), A_n - A > \right\}$$

$$+ o(1), \quad \lim_{n\to\infty} o(1) = 0.$$

So, it follows from 3.2.4 that $\lim_{n\to\infty} \| A_n - A \|_{L^{1,2}} = 0$, and consequently, $A_n \to A$ strongly in L^4.

(2) $\lim_{n\to\infty} \| \phi_n - \phi \|_{L^{1,2}} = 0$.

(3.15)

$$|| \nabla^0 \phi_n - \nabla^0 \phi ||^2_{L^2} = \overbrace{\int_X < \nabla^0 \phi_n, \nabla^0 (\phi_n - \phi) >}^{(1)} - \overbrace{\int_X < \nabla^0 \phi, \nabla^0 (\phi_n - \phi) >}^{(2)}$$

The term (1) leads to

$$\int_X < \nabla^0 \phi_n, \nabla^0 (\phi_n - \phi) >= \int_X < (\nabla^{A_n} - A_n) \phi_n, (\nabla^{A_n} - A_n)(\phi_n - \phi) >=$$

$$\int_X < \nabla^{A_n} \phi_n, \nabla^{A_n}(\phi_n - \phi) > - \int_X < \nabla^{A_n} \phi_n, A_n(\phi_n - \phi) > -$$

$$\int_X < A_n \phi_n, \nabla^{A_n}(\phi_n - \phi) > + \int_X < A_n \phi_n, A_n(\phi_n - \phi) >=$$

$$= \overbrace{d(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}(\phi_n - \phi) - \int_X \frac{|\phi_n|^2 + k_g}{4} < \phi_n, \phi_n - \phi >}^{(11)} -$$

$$\overbrace{\int_X < \nabla^{A_n} \phi_n, A_n(\phi_n - \phi) >}^{(12)} - \overbrace{\int_X < A_n \phi_n, \nabla^{A_n}(\phi_n - \phi) >}^{(13)} +$$

$$+ \overbrace{\int_X < A_n \phi_n, A_n(\phi_n - \phi) >}^{(14)}.$$

The term (2) in 3.15 leads to similar terms named (21), (22), (23) and (24). Let's analyse each one of the overbraced terms obtained above:

(a) terms **(11)** and **(21)**.

$$d(\mathcal{SW}_\alpha)_{(A_n,\phi_n)}(\phi_n - \phi) - \int_X \frac{|\phi_n|^2 + k_g}{4} < \phi_n, \phi_n - \phi > + o(1) =$$

$$< \sigma, \phi_n - \phi > - \int_X \frac{|\phi_n|^2 + k_g}{4} |\phi_n - \phi|^2 - \int_X \frac{|\phi_n|^2 + k_g}{4} < \phi, \phi_n - \phi > +$$

$$o(1) \leq < \sigma, \phi_n - \phi > - \int_X \frac{|\phi_n|^2 + k_g}{4} < \phi, \phi_n - \phi > + o(1)$$

$$\leq || \sigma ||^2_{L^2} \cdot || \phi_n - \phi ||^2_{L^2} + || \frac{|\phi_n|^2 + k_g}{4} ||^2_{L^2} \cdot || \phi ||_\infty \cdot || \phi_n - \phi ||^2_{L^2} + o(1),$$

where $\lim_{n \to \infty} o(1) = 0$. By the similarity among (11) and (21), we conclude the boundeness of term (22).

(b) terms **(12)** and **(22)**.

$i.$ (12)

$$\int_X < \nabla^{A_n}\phi_n, A_n(\phi_n - \phi) > =$$

$$\int_X < \nabla^{A_n}\phi_n, (A_n - A)(\phi_n - \phi) > + \int_X < \nabla^{A_n}\phi_n, A(\phi_n - \phi) >$$

$$\leq \int_X | \nabla^{A_n}\phi_n |^2 \cdot \int_X | A_n - A |^4 \cdot \int_X | \phi_n - \phi |^4 +$$

$$\int | \nabla^{A_n}\phi_n |^2 \cdot \int_X | A(\phi_n - \phi) |^2$$

$ii.$ (21)

$$\int_X < \nabla^A \phi, A(\phi_n - \phi) > \quad \leq \int_X | \nabla^A \phi |^2 \cdot \int_X | A(\phi_n - \phi) |^2$$

The term $\int_X | \nabla^A \phi |^2$ is bounded by 4.0.6 and $A \in C^0$ by 4.0.9.

(c) term $\{(13) - (23)\}$.

$$\int_X < A_n\phi_n, \nabla^{A_n}(\phi_n - \phi) > - \int_X < A\phi, \nabla^A(\phi_n - \phi) > =$$

$$\int_X < (A_n - A)\phi_n, \nabla^{A_n}(\phi_n - \phi) > + \overbrace{\int_X < A\phi_n, \nabla^{A_n}(\phi_n - \phi) >}^{(i)} -$$

$$\int_X < (A_n - A)\phi, \nabla^A(\phi_n - \phi) > - \overbrace{\int_X < A_n\phi, \nabla^A(\phi_n - \phi) >}^{(ii)} =$$

In each of the last two lines above, the first terms are bounded by $\| A_n - A \|_{L^4}$, while the term $\{(i) - (ii)\}$ can be written as

$$\int_X < (A - A_n)\phi_n, \nabla^{A_n}(\phi_n - \phi) > + \int_X < A_n(\phi_n - \phi), \nabla^{A_n}(\phi_n - \phi) > +$$

$$\int_X < A_n\phi, (\overbrace{\nabla^{A_n} - \nabla^A}^{(A_n - A)})(\phi_n - \phi) > .$$

So, it is also bounded by $\| A_n - A \|_{L^4}$.

(d) term $\{(14) - (24)\}$.

$$\int_X < A_n\phi_n, A_n(\phi_n - \phi) > - \int_X < A\phi, A(\phi_n - \phi) > =$$

$$= \int_X < A_n\phi_n, (A_n - A)(\phi_n - \phi) > + \int_X < (A_n - A)\phi_n, A((\phi_n - \phi) > +$$

$$\int | A(\phi_n - \phi) |^2$$

Since $A \in C^0$, it follows that $\lim_{n \to \infty} \| A(\phi_n - \phi) \|^2 = 0$.

$\square$

4. Regularity of the Solution (A, ϕ)

Let $\beta = \{e_i; 1 \leq i \leq 4\}$ be a orthonormal frame fixed on TX with the following properties; for all $i, j \in \{1, 2, 3, 4\}$

(1) (commuting) $[e_i, e_j] = 0$,
(2) $\nabla_{e_i} e_j = 0$ ($\nabla = $ Levi-Civita connection on X).

Let $\beta^* = \{dx_1, \ldots, dx_n\}$ be the dual frame induced on $\mathcal{S}_\alpha{}^*$. From the 2nd-property of the frame β, it follows that $\nabla_{e_i} dx^j = 0$ for all $i, j \in \{1, 2, 3, 4\}$. For the sake of simplicity, let $\nabla_{e_i}^A = \nabla_i^A$. Therefore, $\nabla^A : \Omega^0(ad(\mathfrak{u}_1)) \rightarrow \Omega^1(ad(\mathfrak{u}_1))$ is given by

$$\nabla^A \phi = \sum_l (\nabla_l^A \phi) dx_l \quad \Rightarrow \quad |\nabla^A \phi|^2 = \sum_l |\nabla_l^A \phi|^2,$$

and

$$(\nabla^A)^2 = \sum_{k,l} (\nabla_k^A \nabla_l^A \phi) dx_l \wedge dx_k \quad \Rightarrow \quad |(\nabla^A)^2|^2 = \sum_{k,l} |\nabla_k^A \nabla_l^A \phi|^2.$$

In this setting, the 2-form of curvature of the connection A is given by

$$(F_A)_{kl} = F_{kl} = \nabla_l^A \nabla_k^A - \nabla_k^A \nabla_l^A.$$

In order to compute the operator $\Delta_A = (\nabla^A)^* \nabla^A : \Omega^0(\mathcal{S}_\alpha^+) \rightarrow \Omega^0(\mathcal{S}_\alpha^+)$, let $* : \Omega^i(\mathcal{S}_\alpha) \rightarrow \Omega^{4-i}(\mathcal{S}_\alpha)$ be the Hodge operator and consider the identity

$$(\nabla^A)^* = - * \nabla^A * : \Omega^1(\mathcal{S}_\alpha^+) \rightarrow \Omega^0(\mathcal{S}_\alpha^+).$$

Hence,

$$\Delta_A \phi = - \sum_k \nabla_k^A \nabla_k^A \phi.$$

In this way,

(4.1)
$$|\Delta_A \phi|^2 = \sum_{k,l} < \nabla_k^A \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi > =$$

$$= \sum_{k,l} \left[\nabla_k^A (< \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi >) - < \nabla_k^A \phi, \nabla_k^A \nabla_l^A \nabla_l^A \phi > \right] =$$

$$= \sum_{k,l} \left[\nabla_k^A (< \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi >) - < \nabla_k^A \phi, \nabla_l^A \nabla_k^A \nabla_l^A \phi > - < \nabla_k^A \phi, F_{lk} \nabla_l^A \phi > \right]$$

$$= \sum_{k,l} \left[\nabla_k^A (< \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi >) - \nabla_l^A (< \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi >) \right] +$$

$$+ \sum_{k,l} \left[< \nabla_l^A \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi > + < \nabla_k^A \phi, F_{lk} \nabla_l^A \phi > \right] =$$

(4.2)
$$= \sum_{k,l} \left[\nabla_k^A (< \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi >) - \nabla_l^A (< \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi >) \right] + \sum_{k,l} | \nabla_k^A \nabla_l^A \phi |^2 +$$
$$+ \sum_{k,l} \left[< F_{kl} \phi, \nabla_k^A \nabla_l^A \phi > + < \nabla_k^A \phi, F_{kl} \nabla_l^A \phi > \right]$$

and so,

$$| (\nabla^A)^2 \phi |^2 \leq \quad | \Delta_A \phi |^2 + \sum_{k,l} \left\{ | \nabla_k^A (< \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi >) | \right\} +$$

(4.3)
$$\sum_{k,l} \left\{ | \nabla_l^A (< \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi >) | \right\} + \sum_{k,l} \left\{ | < F_{kl} \phi, \nabla_k^A \phi \nabla_l^A \phi > | \right\} +$$
$$\sum_{k,l} \left\{ | < \nabla_k^A \phi, F_{kl} \nabla_l^A \phi > | \right\}$$

Now, by applying the inequalities

$$\left(\sum_i a_i \right)^r \leq K_r . \sum_i | a_i |^r, \quad \sqrt{\sum_{i=1}^n a_i} \leq \sum_{i=1}^n \sqrt{a_i}$$

to 4.3, we get

$$| (\nabla^A)^2 \phi |^p \leq \quad K_p . | \Delta_A \phi |^p + K_p . \sum_{k,l} \left\{ | \nabla_k^A (< \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi >) |^{\frac{p}{2}} \right\} +$$
$$K_p \sum_{k,l} \left\{ | \nabla_l^A (< \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi >) |^{\frac{p}{2}} \right\} + \sum_{k,l} \left\{ | < F_{kl} \phi, \nabla_k^A \phi \nabla_l^A \phi > |^{\frac{p}{2}} \right\} +$$
$$\sum_{k,l} \left\{ | < \nabla_k^A \phi, F_{kl} \nabla_l^A \phi > |^{\frac{p}{2}} \right\} ;$$

After integrating, it follows that

$$k_1 . \| (\nabla^A)^2 \phi \|_{L^p}^p \quad \leq \quad \| \Delta_A \phi \|_{L^p}^p + k_2 . \| \nabla^A \phi \|_{L^p}^p + + k_3 . \| F_A(\phi) \|_{L^p}^p +$$
$$+ k_4 . \| F_A(\nabla^A \phi) \|_{L^p}^p + k_5 . \sum_{k,l} \int_x \left\{ | \nabla_k^A (< \nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi >) |^{\frac{p}{2}} \right\} +$$
(4.4)
$$+ k_6 \sum_{k,l} \int_X \left\{ | \nabla_l^A (< \nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi >) |^{\frac{p}{2}} \right\}$$

The boundness of the right hand side of 4.4 results from the analysis of each term;

Proposition 4.0.6. *Let* $(A, \phi) \in \mathcal{C}_\alpha$ *be a solution of equations in (1.4). If* $\sigma \in L^\infty$*, then*

    (1) $\nabla^A \phi \in L^2$,
    (2) $\Delta_A \phi \in L^2$.

Proof.   (1) $\nabla^A \phi \in L^2$

$$< \Delta_A \phi, \phi > + \left(\frac{\mid \phi \mid^2 + k_g}{4} \right) \mid \phi \mid^2 = < \sigma, \phi >$$

$$\Rightarrow \quad \mid \nabla^A \phi \mid^2 + \left(\frac{\mid \phi \mid^2 + k_g}{4} \right) \mid \phi \mid^2 = < \sigma, \phi > \quad \leq$$

$$\leq \quad \frac{1}{\epsilon^2} \mid \sigma \mid^2 + \epsilon^2 \mid \phi \mid^2$$

Therefore,

$$\mid \nabla^A \phi \mid^2 < \frac{1}{\epsilon^2} \mid \sigma \mid^2 + (\epsilon^2 - \frac{k_g}{4}) \mid \phi \mid^2 - \frac{\mid \phi \mid^4}{4}$$

From 3.2.3, there exists a polynomial p, which coefficients depend on (X, g) and ϵ, such that

(4.5)
$$\|\nabla^A \phi\|_{L^2}^2 < p(\|\sigma\|_\infty)$$

So, $\nabla^A \phi \in L^2$.
(2) $\Delta_A \phi \in L^2$.

$$< \Delta_A \phi, \Delta_A \phi > + \frac{\mid \phi \mid^2 + k_g}{4} < \phi, \Delta_A \phi > = < \sigma, \Delta_A \phi >$$

let $0 < \epsilon < 1$,

$$\mid \Delta_A \phi \mid^2 + \frac{\mid \phi \mid^2 + k_g}{4} \mid \nabla^A \phi \mid^2 = < \sigma, \Delta_A \phi > \quad <$$

$$< \frac{1}{\epsilon^2} \mid \sigma \mid^2 + \epsilon^2 \mid \Delta_A \phi \mid^2$$

(4.6)
$$(1 - \epsilon^2) \mid \Delta_A \phi \mid^2 + \frac{\mid \phi \mid^2 + k_g}{4} \mid \nabla^A \phi \mid^2 < \frac{1}{\epsilon^2} \mid \sigma \mid^2$$

By the boundness of the term

(4.7)
$$\int_X \mid \phi \mid^2 \cdot \mid \nabla^A \phi \mid^2 < \|\phi\|_\infty^2 \cdot \|\nabla^A \phi\|_{L^2}^2,$$

it follows the existence of a polynomial q, which coefficients depending on ϵ and (X, g), such that

(4.8)
$$\|\Delta_A \phi\|_{L^2} < q(\|\sigma\|_\infty).$$

$\square$

Proposition 4.0.7. *Let (A, ϕ) be solutions of the $\mathcal{SW}_\alpha$-equations where $(\Theta, \sigma) \in L^{1,2} \times (L^{1,2} \cap L^\infty)$, then $F_A \in L^q$, for all $q < \infty$.*

Proof. By 1.5, $\Phi^*(\nabla^A \phi) = \frac{1}{2} \nabla^A(|\phi|^2)$, and so,

$$d^* F_A + 4\Phi^*(\nabla^A \phi) = \Theta \quad \Rightarrow \quad \| d^* F_A \|_{L^2}^2 \leq \| \phi \|_{L^{1,2}}^2 + \| \Theta \|_{L^2}$$

There are two cases to be analysed;

(1) F_A is harmonic.
   Since the Laplacian defined on $\mathfrak{u}_1$-forms is a elliptic operator, the fundamental inequality for elliptic operators claims that there exists a constant C_k such that

(4.9) $$\| F_A \|_{L^{k+2,2}} \leq \| \Delta F_A \|_{L^{k,2}} + C_k \| F_A \|_{L^2} .$$

   Consequently, F_A being harmonic implies, for all $k \in \mathbb{N}$, that

$$\| F_A \|_{L^{k,2}} \leq C_k \| F_A \|_{L^2}, \quad \Rightarrow \quad F_A \in C^\infty.$$

(2) F_A is not harmonic.
   In this case, since $\Theta \in L^{1,2}$, $\phi \in L^\infty$ and

$$\Delta_A F_A = d(<\phi, \nabla^A \phi>) + d\Theta = <\phi, F_A(\phi)> + d\Theta,$$

   it follows that $F_A \in L^{2,2}$.
   Therefore, by the Sobolev embedding theorem $F_A \in L^q$, for all $q < \infty$.
   $\square$

Proposition 4.0.8. *Let (A, ϕ) be solutions of the $\mathcal{SW}_\alpha$-equations where $(\Theta, \sigma) \in L^{1,2} \times (L^{1,2} \cap L^\infty)$, then $(\nabla^A)^2 \phi \in L^p$, for all $1 < p < 2$.*

Proof. In 4.4, we must take care of the last terms;

(1) $F(\nabla^A \phi) \in L^p$, for all $1 < p < 2$. By Young's inequality,

$$\| F(\nabla^A \phi) \|_{L^p} \leq \| F_A \|_{L^{\frac{2p}{2-p}}} \cdot \| \nabla^A \phi \|_{L^2} .$$

(2) There is no contribution from the divergent terms, since

$$\int_x \left\{ | \nabla_k^A (<\nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi>) |^{\frac{p}{2}} \right\} \leq [vol(X)]^{\frac{2-p}{p}} \int_x \left\{ | \nabla_k^A (<\nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi>) | \right\} .$$

   In the same way,

$$\sum_{k,l} \int_x \left\{ | \nabla_k^A (<\nabla_k^A \phi, \nabla_l^A \nabla_l^A \phi>) |^{\frac{p}{2}} \right\} = 0$$

$$\sum_{k,l} \int_X \left\{ | \nabla_l^A (<\nabla_k^A \phi, \nabla_k^A \nabla_l^A \phi>) |^{\frac{p}{2}} \right\} = 0.$$

   The estimates above applied to 4.4 implies that

$$\| (\nabla^A)^2 \phi \|_{L^p} \leq k_1 \| \Delta_A \phi \|_{L^p}^p + k_2 \| \nabla^A \phi \|_{L^p}^p + k_3 \| \nabla^A \phi \|_{L^p}^p +$$
$$+ k_4 \| F_A(\phi) \|_{L^p}^p + k_5 \| F_A \|_{L^{\frac{p}{2-p}}} \cdot \| \nabla^A \phi \|_{L^p}^p$$

   $\square$

Thus, $\phi \in L^{2,p}$, for all $1 < p < 2$. Considering that $\sigma \in L^{1,2}$, the bootstrap argument applied on 1.4 implies that $\phi \in L^{3,p}$, for every $k \geq 2$ and $1 < p < 2$. Hence, by Sobolev embedding theorem, $\phi \in C^0$.

Theorem 4.0.9. *Let (A, ϕ) be a solution of the $\mathcal{SW}_\alpha$-equations where $(\Theta, \sigma) \in L^{k,2}(\Omega^1(ad(\mathfrak{u}_1))) \oplus (L^{k,2}(\Gamma(\mathcal{S}_\alpha^+)) \cap L^\infty(\Gamma(\mathcal{S}_\alpha^+)))$, then $(A, \phi) \in L^{k+2,p} \times (L^{k+2,2} \cap L^\infty)$, for all $1 < p < 2$. Moreover, if $k > 2$, then $(A, \phi) \in C^r \times C^r$, for all $r < k$.*

Proof. (1) If $\Theta \in L^{k,2}$, then by 4.0.7 $F_A \in L^{k+1,2}$. Consequently, by 2.1.2, $A \in L^{k+2,2}$.

(2) The Sobolev class of ϕ is obtained by the bootstrap argument.

$\square$

APPENDIX A. ESTIMATES FOR SOLUTIONS OF 3rd-DEGREE EQUATION

Let $p, q \in \mathbb{R}$ and consider the equation

$$x^3 + px + q = 0 \tag{A.1}$$

Proposition A.0.10. *The solutions of (A.1) are given in [8] by*

$$x_1 = z_1 + z_2, \quad x_2 = z_1 + \lambda z_2 \quad and \quad y_3 = z_1 + \lambda^2 z_2, \tag{A.2}$$

where

$$z_1 = \sqrt[3]{-\frac{q}{2} + \sqrt[2]{D}} \quad z_2 = \sqrt[3]{-\frac{q}{2} - \sqrt[2]{D}},$$

$$D = \frac{p^3}{27} + \frac{q^2}{4},$$

and $\lambda \in \mathbb{C}$ satisfies $\lambda^3 = 1$.

Corollary A.0.11. *Let $q, p \in \mathbb{R}$ such that $q < 0$ and $p < 0$. So, the solutions of equation (A.1) are estimates according with the following cases;*

(1) $D \geq 0$

$$\mid x_i \mid \leq \frac{8}{3} + \frac{1}{3} \mid q \mid + \frac{1}{12} q^2 + \frac{1}{81} p^3 \tag{A.3}$$

(2) $D < 0$

$$\mid x_i \mid \leq 3 + \frac{1}{6} q^2 + \frac{1}{81} \mid p \mid^3 \tag{A.4}$$

Proof. Since

$$\mid x_i \mid \leq \mid z_1 \mid + \mid z_2 \mid$$

it is enough to estimate $\mid z_1 \mid$ and $\mid z_2 \mid$. The basics identity needed are the following: suppose $x \geq 0$, so

$$\sqrt[2]{x} \leq 1 + \frac{1}{2} x$$

$$\sqrt[3]{x} \leq 1 + \frac{1}{3} x$$

(1) $D \geq 0$

In this case, $z_1, z_2 \in \mathbb{R}$ and

$$\mid z_1 \mid = \sqrt[3]{\mid -\frac{q}{2} + \sqrt[2]{D} \mid} \leq 1 + \frac{1}{3} \mid -\frac{q}{2} + \sqrt[2]{D} \mid \leq \frac{4}{3} + \frac{1}{6} \mid q \mid + \frac{1}{6} D$$

So,

$$\mid z_1 \mid \leq \frac{4}{3} + \frac{1}{6} \mid q \mid + \frac{1}{24} q^2 + \frac{1}{162} p^3$$

The same estimate can be obtained for $\mid z_2 \mid$. Hence,

$$\mid x_i \mid \leq \frac{8}{3} + \frac{1}{3} \mid q \mid + \frac{1}{12} q^2 + \frac{1}{81} p^3$$

(2) $D \leq 0$

In this case, $z_1, z_2 \in \mathbb{C} - \mathbb{R}$. Since $D \in \mathbb{R}$, we can write $\sqrt[2]{D} = i \sqrt[2]{\mid D \mid}$ and

$$z_1 = \sqrt[3]{-\frac{1}{2}q + i\sqrt[2]{D}}, \quad z_2 = \sqrt[3]{-\frac{1}{2}q - i\sqrt[2]{D}}$$

Therefore,

$$\mid z_i \mid^2 = \sqrt[3]{\frac{q^2}{4} + \mid D \mid} < 1 + \frac{1}{12} q^2 + \frac{1}{3} \mid D \mid \leq 1 + \frac{1}{6} q^2 + \frac{1}{81} \mid p \mid^3$$

and

$$\mid z_i \mid < \frac{3}{2} + \frac{1}{12} q^2 + \frac{1}{162} \mid p \mid^3$$

Hence,

$$\mid x_i \mid < 3 + \frac{1}{6} q^2 + \frac{1}{81} \mid p \mid^3$$

$\square$

REFERENCES

[1] ATIYAH, M; JONES, J.: *Topological Aspects of Yang-Mills Theory, Comm.Math.Physics* **61** (1979), 97-118.

[2] DORIA, C.M.: *The Homotopy Type of the Seiberg-Witten Configuration Space*, pre-print 2001.

[3] DONALDSON, S.K.: *The Seiberg-Witten Equations and 4-Manifold Topology*, Bull.Am.Math.Soc., New Ser. **33**, n°1 (1996), 45-70.

[4] DONALDSON, S.K.; KRONHEIMER, P.: *The Geometry of 4-Manifold*, Oxford University Press, 1991.

[5] EELLS, J.; LEMAIRE, L.: *Selected Topics in Harmonic Maps*, CBMS n^o**50**, AMS, 1980.

[6] FREED, D.; UHLENBECK, K.: *Instantons and Four Manifolds*, MSRI Publications, Vol 1, Springer-Verlag, 1984

[7] GILBARG, D.; TRUDINGER, N.S.: *Elliptic Partial Differential Equations of Second Order*, 2nd-edition, SCSM 224, Springer-Verlag, 1983.

[8] GONÇALVES, A.; *Introdução à Álgebra* , Projeto Euclides, SBM

[9] JOST, J.; PENG, X.; WANG, G.: *Variational Aspects of the Seiberg-Witten Functional*, Calculus of Variation **4** (1996) , 205-218.

[10] KATO, T.: *Perturbation Theory for Linear Operators*, 2nd-edition, SCSM 132, Springer-Verlag, 1984

[11] LAWSON, H.B.; MICHELSON, M.L.: *Spin Geometry*, Princeton University Press, 1989.

[12] MARINI, A.; *Dirichlet and Neumenn Boundary Value Problems for Yang-Mills Connections*, Commuc. on Pure and Applied Math, **XLV** (1992), 1015-1050.

[13] MORGAN, J.: *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*, Math. Notes **44**, Princeton Press.

[14] PALAIS, R.S.: *Foundations of Global Non-Linear Analysis*, Benjamin, inc, 1968.

[15] UHLENBECK, K.: *Connections with L^p bounds on Curvature*, Comm. Math. Phys. **83**, 1982, pp 31-42.

Universidade Federal de Santa Catarina
Campus Universitario , Trindade
Florianópolis - SC , Brasil
CEP: 88.040-900
http://www.mtm.ufsc.br

E-mail address: `cmdoria@mtm.ufsc.br`